Debt Overhang, Risk Shifting, and Zombie Lending

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Abstract

After bubbles collapse, banks have often rolled over debt at subsidized rates to indebted borrowers or "zombie firms." This paper explores the incentives to restructure debt in a game where the potential diversion of funds and risk shifting cause debt overhang. We provide conditions under which it is privately optimal to zombie lend even when it is socially inefficient. When a firm becomes over-indebted, the firm loses access to competitive funding and its bank can exert monopoly power. The bank prefers to zombie lend given that providing funds for investment is not profitable due to risk shifting, while liquidation entails costs. The model explains the ineffectiveness of traditional policies in the presence of zombies such as bank recapitalization and monetary policy and highlights the necessity of debt haircuts, which are not privately optimal.

JEL Classification: G21, G28, G32, G33.

Keywords: zombie lending, debt overhang, risk shifting, bank–firm relationship, monetary policy, debt haircut.

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1 Introduction

Sharp drops in asset prices can wipe out collateral value and spell trouble for both banks and firms. Insolvent firms in a situation of debt overhang may continue to operate if banks roll over their debt at subsidized rates, a practice known as zombie-lending. These "zombie firms" are kept afloat but fail to grow. This phenomenon has substantial negative effects on investment and productivity. In the face of debt overhang, decentralized bargaining should theoretically lead to a welfare-improving situation, either a liquidated firm or to debt restructuring. However, the experiences of Japan during the 1990s and of Europe during the 2010s show that there can be substantial delays in this recovery (see Figure 1).

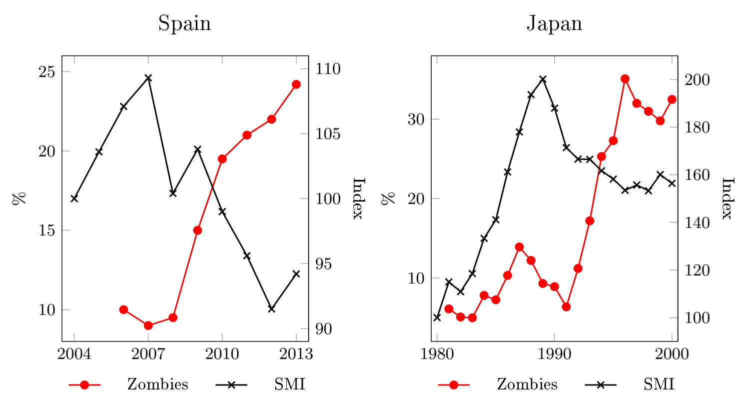


Figure 1: Percentage of Zombie Firms (left axis) and Stock Market Index (SMI, right axis). Zombies are firms that receive below-market rates. Sources: Aragon (2018) for Spain and Caballero et al. (2008) for Japan.

This paper develops a model of bank-firm interaction to explain how zombie firms arise in decentralized equilibrium and fail to be liquidated, even when restructuring is formally allowed and banks are capitalized. Zombie-lending is defined in the paper as the practice of continuously lending for survival to a financially insolvent firm, the zombie. Previous studies of debt forbearance have focused on regulatory forbearance of a troubled bank or on informational frictions (see Related Literature). The model generates the phenomenon in a full information setting where firms with operational profits but financial losses have access to solvent banks and profitable projects. The policy implications are in stark contrast to previous studies which assume that firms are operationally unprofitable and therefore restoring efficiency requires the liquidation of their assets.

The result of zombie lending as an equilibrium strategy emerges from the disconnect between the incentives of the bank and the firm. When debt is excessively high, the firm cannot access the competitive credit market and is thus locked in with the incumbent bank. The bank can then extract a large surplus from the captive firm without funding any investment. Alternatively, if the bank were to restructure and extend a loan to enact an efficient project, the required repayment would need to be large. The firm would then, overburdened, either divert the funds or shift the risk of the investment project.

The bank anticipates the firm's moral hazard on project choice and thus prefers to take the firm's business-as-usual revenue stream as payment. Under risk shifting, the expected return of extending fresh funds or allowing the company to keep enough funds from its operating profits to self-fund investment is low. The bank decides to ignore previously sunk debt and take the firm's profits as interest without ever liquidating or restructuring it, turning the firm into a zombie.

The model presents a game among three agents, a firm and two banks, that lasts three periods. The firm has a revenue stream with its business-as-usual technology and an exogenous level of debt contracted with one of the banks, the incumbent. The firm can invest in one project chosen from a risk continuum that is non-contractible. If successful, the project allows the firm to increase its revenue stream. When initial debt is low, the firm can repay its incumbent bank and borrow from either of the two banks to finance investment. Bank competition pushes the cost of borrowing until banks make zero profits. The firm invests in a project and social welfare is maximized. As debt increases, the economy is pushed into an intermediate zone where there is an inefficient level of risk but the socially beneficial project is still carried out.

However, when the firm suddenly becomes insolvent, a hold-up problem arises. During "crisis times," debt becomes high and the firm enters into debt overhang. Because the bank anticipates the possibility of risk shifting and fund diversion, no bank is willing to lend and the firm loses access to the competitive market. Thus, the incumbent bank becomes a monopolist and, as such, has two broad strategies in the first period: it can (i) liquidate the firm early and put the proceeds in a risk-free technology, or (ii) restructure. Restructuring can imply extracting all the operating profits from the firm, zombie-lending, or allowing the firm enough funds to invest. The firm, in turn, can use the funds at its own discretion in the intermediate period. At the end of the game, the bank is either repaid or liquidates the firm. In this model, we demonstrate that there are situations under which the bank decides to keep the firm under survival lending even without hopes of "resurrection." The reason is that the moral hazard imposes an upper limit on the upside, but leaves the bank willing to take as much as it can from the business-as-usual stream. Importantly, the bank can zombie-lend even when the firm has a profitable project or when it is socially beneficial to restructure debt.

Section 3.6 extends the model to the case of a financially distressed bank and firm simultaneously. In this case, the bank's assets are subject to a shock that may render it bankrupt. If the bank decides to liquidate the firm early, it must acknowledge the loss and will thus be in a more fragile position to face the shock. A bank in a strong capital condition can bear the loss and liquidate the firm. In contrast, a weak bank, fearing the shock, may be unwilling to do so and would keep the firm afloat.

The analysis shows that the conditions for zombie-lending and the effects of policy depend on whether the firm has positive operational profits or not. Therefore, the firm's ability to profit independently of its financial burden impacts the outcomes of policy interventions. A larger and safer business-as-usual revenue stream and a more expensive project widen the region for zombie-lending. On the other hand, firms with operational losses will become zombies only if banks are financially distressed, as otherwise they would be liquidated.

The model yields several policy prescriptions. First, partial debt forgiveness in the form of a debt haircut increases welfare but is not privately optimal. The bank is unwilling to privately reduce the debt burden because the business-as-usual income stream from the firm is, given the possibility of risk shifting, higher than the expected return under the renegotiated plan. A haircut on debt decreases the bank's market power, allowing the firm to regain access to the competitive market and invest. Social surplus increases and redistribution can improve the bank's welfare as well. Second, recapitalization is irrelevant on the margin that determines investment. Recapitalizing a bank changes only its incentive to liquidate, and liquidation is dominated for viable firms. Capital matters only for failing firms, where an undercapitalized bank sustains those with operating losses. Even in that case, it does not lead to higher investment. Third, when dealing with zombie firms, monetary policy is ineffective. The repayment the monopolist demands is independent of the interest rate, and thus lowering it does not affect the firm's effective burden. Tightening can push firms into zombie territory, but loosening cannot pull them back out.

Related Literature

Zombie firms were first identified after the burst of the Japanese asset price bubble, and the country's growth has remained stagnant for decades since. Similar dynamics were observed after the European debt crisis, and several facts are common to both episodes. First, companies that receive survival lending are highly indebted, and their prevalence is higher after the burst of a bubble (Caballero et al., 2008; Hoshi and Kashyap, 2004; Andrews and Petroulakis, 2019;

Aragon, 2018). Second, these firms are more likely to be in a lending relationship with a financially distressed bank (Peek and Rosengren, 2005), particularly when restructuring is costly (Andrews and Petroulakis, 2019). Third, these firms are less likely to invest (Peek and Rosengren, 2005; Kalemli-Ozcan et al., 2018). Finally, when zombies are widespread, recapitalizing banks and lowering the interest rate have had mild or nil effects on reigniting the economy (Schivardi et al., 2017; Acharya et al., 2016; Paligorova and Santos, 2017; Berger et al., 2016; Hoshi and Kashyap, 2004). We contribute to this literature by showing how this phenomenon can arise in a decentralized equilibrium and accounting for these regularities by focusing on the incentives of both the firm and the bank.

Zombie-lending is, in essence, the study of debt forbearance, and, as such, lies at the intersection between the literature of debt overhang and renegotiation. The debt overhang literature starting with Myers (1977) highlights that large amounts of debt lead to under-investment. Such work spans a large literature focusing mostly on the incentives of borrowers. We, instead, focus on the incentives of both borrowers and lenders. Kovrijnykh and Szentes (2007) focus on both actors as they study countries in debt overhang. We simplify their setup and apply it to firms which, unlike countries, can be liquidated and have a scope for shifting risk. These two differences change the space of strategies and outcomes.

The literature on renegotiation[1] highlights that, when there is a bargaining surplus, default is inefficient and renegotiation mitigates the under-investment result arising from debt overhang. As such, our paper relates to delays in this process. Admati and Perry (1987), Vilanova (2004), and Kahl (2002) explain these delays based on information frictions and relative bargaining powers between borrowers and lenders. In our model, shocks are permanent and the evolution of the firm's output is known by both parties. The delay here arises because the bank does not find it profitable to renegotiate at any point, as it can take the totality of the firm's business-as-usual profits, and if it did decide to lend further funds, the firm would increase the risk, thus rendering it unprofitable.

The literature on debt forbearance has usually focused on the incentives of the regulators (Hellwig et al., 2012; Bruche and Llobet, 2013; Aghion et al., 1999) or, in the case of state-owned companies, the government (Berglof and Roland, 1998; Aghion and Bolton, 1992). The literature focuses on regulatory schemes where the bank can hide a bad loan from its principal. We, instead, abstract from the incentives of the regulators to focus solely on profit-seeking agents to show how competitive markets can lead to the same situation, and explore macroeconomic policy under this condition. We show that regulatory capture is not necessary

[1] See for example Hart and Moore (1998); Hellwig (1977); Hart and Tirole (1988); Frantz and Instefjord (2019); Favara et al. (2017); Pawlina (2010).

for the existence of these firms. An exception focusing on forbearance from the point of view of the lenders is Rajan (1994). In their mechanism, a banker with reputational concerns decides to "gamble for resurrection" when faced with an unprofitable firm. Hu and Varas (2021) provide a theory of zombie lending and is closest to our paper. Their theory relies on information production, information acquisition, and reputation formation. An entrepreneur with a good reputation will be able to roll over loans even when insolvent. The theory put forward in our paper complements this explanation by focusing on debt overhang and monopolistic power under full information. Whereas explanations based on informational frictions normally imply a constrained efficient solution, under the theory in this paper, moral hazard and risk shifting interact to prevent the renegotiation outcome from arising with crucial implications for policy. Acharya et al. (2021) recently study zombie lending as a policy trap of regulatory forbearance aiming to increase credit during the zero lower bound. We need neither fragile banks nor regulatory will, and see our paper as a complementary explanation of debt forbearance.

Section 2 presents the model and Section 3 the main result. The case of an undercapitalized bank is in Section 3.6. Policy is discussed in Section 4. Section 5 concludes and discusses the results and assumptions.

2 Model

In this section we present the model environment and establish two preliminary results that are used in the remainder of the paper. Section 2.2 solves the firm's project choice and shows that risk shifting places a limit on what any lender can recover. Section 2.3 shows that this ceiling, together with the assumption of seniority of inherited debt, determines the willingness to lend from competitor banks. Thus, debt overhang emerges as a region of the parameter space rather than an assumption. In Section 2.4 we show that depending on the inherited debt level, the firm may find itself in one of three regions, which differ in their efficiency. Section 2.5 focuses on the region in which the mechanism of the paper operates.

2.1 Environment

There are three risk-neutral agents: a firm, an incumbent bank, and a competitor bank. They live for three periods: $t \in \{0, 1, 2\}$. The firm has a business-as-usual technology (BAU) and an inherited level of debt with the incumbent. It also has access to a range of projects that entail risk. However, the project choice is non-contractible, leading to moral hazard. At $t = 0$ the two banks simultaneously make offers and the firm accepts one or is liquidated. At $t = 1$ the firm produces, settles its obligations with the bank, and decides privately whether and how to invest. At $t = 2$ the outcome is realised and the firm settles its remaining obligation. Figure 2 summarises the sequence. All parameters are common knowledge.

The firm operates its BAU technology yielding an output $y_0 = 1 - p_0$ at $t = 1$, where $p_0 \in (0, 1)$ is fixed. At $t = 2$ the same technology delivers y_0 with probability p_0 and zero otherwise. At $t = 1$ the firm may instead pay a fixed cost $X > 0$ to adopt a new project. These projects are indexed by their success probability $p \in (0, 1)$. A given project with probability p returns

$$y(p) = 1 - p \quad \text{on success,} \qquad 0 \quad \text{on failure.} \tag{1}$$

The expression (1) shows that a riskier project pays more when it succeeds and nothing on failure. If the firm were to pay X to adopt a project, it replaces the BAU technology at $t = 2$.[2]

A firm can be liquidated yielding a recovery value $\ell \in (0, 1)$, which we take as a primitive. The recovery accrues to the senior claimant: the firm's owners when the firm has no debt outstanding, and the creditor otherwise.[3]

At $t = 0$ an unexpected shock scales BAU revenue by $\epsilon \in (0, 1]$, so that effective output is ϵy_0, with $\epsilon = 1$ the pre-shock benchmark. Throughout this section, the firm remains operationally viable: revenue is positive and an efficient project is available. The distress it experiences is therefore purely financial. Section 3.5 treats the case of a firm with operational losses.

Banks and the firm can store cash between $t = 1$ and $t = 2$ at the gross risk-free rate $R > 1$. Every date-1 quantity consequently enters a date-2 payoff multiplied by R.

Frictions. Two of the firm's actions are non-contractible: its choice of project and its use of the funds. The first friction leads to risk shifting: demanding a higher repayment pushes the firm toward riskier projects. This imposes an upper bound on any repayment a lender can earn. Section 2.2 characterizes the bound.

[2] The project's profit is independent of X. Section 2.5 explores the consequences of this feature.

[3] This asymmetry underlies the efficiency result in Section 2.3.

The second friction means that the firm can divert funds. In other words, the firm can retain the cash it holds and put it in a risk-free technology. Therefore, a contract only induces investment if the firm's return from the project exceeds what it would keep by diverting. Retained cash is certain and immediate, while the project is risky and pays only at $t = 2$. Since a firm that diverts X stores the proceeds, the interest rate affects the incentive to divert. The two frictions affect the problem differently: risk shifting limits what a lender can demand and diversion limits what a lender can advance. Their interaction generates zombie lending in Section 3.

Contracts. The incumbent and the competitor bank make an offer (r, b) at $t = 0$, where r is a net date-1 payment from the firm and b a date-2 payment. Alternatively, the banks can offer liquidation, either early or late, i.e., at $t = 1$ or $t = 2$. The liquidation offer is written σ^ℓ. Payments are viewed from the perspective of the firm: $r > 0$ implies the firm pays the bank and $r < 0$ means the bank advances funds. In that case we write $A \equiv -r$ for the amount advanced. Early liquidation, $(\sigma^\ell, 0)$, delivers the bank $R\ell$ at $t = 2$ after being stored; and late liquidation, (r, σ^ℓ), delivers the bank ℓ at $t = 2$ on top of the date-1 payment r, itself worth Rr by then. Section 3.2 characterizes the equilibrium strategies.

The contract space is constrained by the frictions above. A bank would never advance more than the cost of investment, X, since extra cash does not raise the bank's potential recovery. We make the following assumption regarding payments between banks.

Assumption 1 (Payments between banks)**.** A bank may retire another bank's claim through a direct payment, which is not subject to the cap on advances to the firm.

A competitor bank cannot rescue an over-indebted firm by handing it cash, but can buy the incumbent's claim under Assumption 1, as in mortgage refinancing, where the funds do not reach the borrower.

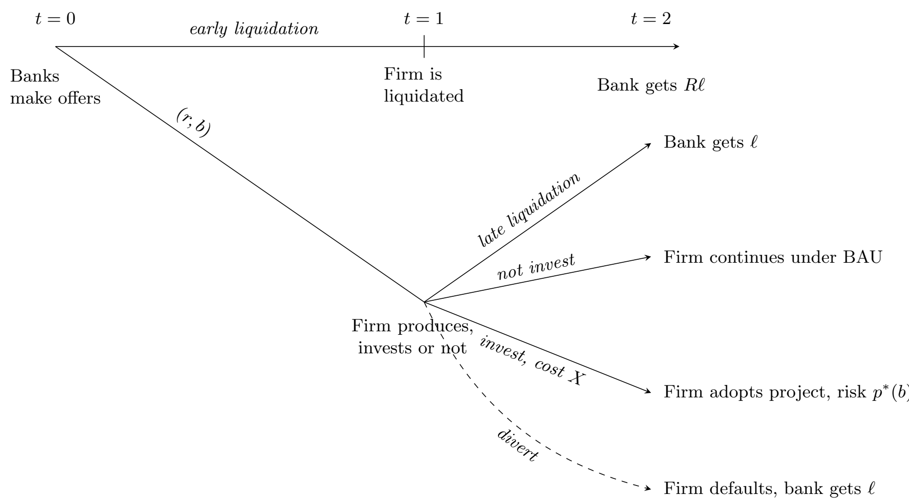


Figure 2: Timing of the model. At $t = 0$ the shock is realised and the two banks make simultaneous offers. If the contract calls for immediate liquidation, the bank collects $R\ell$ at $t = 2$. Otherwise the firm produces at $t = 1$ and makes two non-contractible decisions: whether to pay X and invest or divert it, and, having invested, which project $p^*(b)$ to run. The firm owes b at $t = 2$ and its output succeeds with probability p_0 under the BAU technology and $p^*(b)$ if it invests. In case the project fails, the bank collects ℓ. Late liquidation is worth ℓ to the bank.

2.2 Risk shifting

In this subsection we present a simple risk-shifting mechanism that we embed in the game. The central result is that the lender's expected recovery is a hump-shaped function of the repayment, b. Therefore, the bank's collection at $t = 2$ has a ceiling, which bounds the lending capacity.

A firm that has invested and owes b at $t = 2$ chooses p to solve $\max_p \; p\big(y(p) - b\big)$. Using (1), the objective is strictly concave and the solution is interior for every $b \in (0, 1)$:

$$p^*(b) = \frac{1 - b}{2}. \tag{2}$$

Since $dp^*/db < 0$, a heavier repayment obligation induces a riskier project. Intuitively, a firm owing a large amount prefers to gamble on a project that, if it succeeds, leaves the firm a surplus after repayment to the bank. The repayment the lender demands determines the risk the firm takes.

Substituting (2) into the firm's objective, the two parties' expected date-2 payoffs are

$$U(b) = \frac{(1-b)^2}{4}, \qquad V(b) = p^* b + (1-p^*)\ell = \frac{b(1-b)}{2} + \frac{(1+b)\ell}{2}, \tag{3}$$

where U is the borrower's payoff and V the lender's. The two differ in the failure state: a liquidated firm receives nothing, whereas the lender still collects the recovery ℓ. Social welfare is total date-2 surplus,

$$W(b) \;\equiv\; U(b) + V(b) \;=\; \frac{1-b^2}{4} + \frac{(1+b)\ell}{2}, \tag{4}$$

Note that $U + V = W$ since b is purely a transfer. Both V and W are concave, but they peak at different repayments, and the following lemma locates the two maxima.

Lemma 1 (Two peaks)**.** *V is strictly concave and attains its maximum at*

$$b^M = \frac{1+\ell}{2}, \qquad \textit{with} \qquad p^M \equiv p^*(b^M) = \frac{1-\ell}{4}, \qquad V(b^M) = \frac{1+6\ell+\ell^2}{8}.$$

W attains its maximum at $b = \ell$, where $V(\ell) = \ell$ and $p^*(\ell) = \frac{1-\ell}{2}$. Moreover $b^M > \ell$ and $V(b^M) - \ell = \frac{(1-\ell)^2}{8} > 0$.

Two features of Lemma 1 recur throughout. First, R does not appear in b^M. A lender with market power optimally chooses a repayment set by the borrower's risk response, not by the cost of funds. A lender contemplating a higher repayment faces a tradeoff between a larger payment and a lower probability of receiving it. Second, $b = \ell$ maximises social surplus. Suppose a firm considers a safer project. The firm sets $1 - 2p - b = 0$ and the planner sets $1 - 2p - \ell = 0$. Both value the marginal increase in safety in the same way, $1 - 2p$. The cost, however, does differ: the firm loses b, since a safer project pays the creditor more often, while for society it is ℓ, the recovery forgone when failure becomes less likely.

The wedge is an externality in the failure state. When the project fails the firm receives nothing, but society does not lose everything, because the recovery accrues to the lender. A firm with no debt therefore treats failure as a total loss when it is not socially the case, and over-corrects to avoid it, choosing too safe of a project. Leverage, b, makes success worth less and pushes the firm back towards higher risk. At $b = \ell$ the correction exactly offsets the externality. A firm with no debt keeps ℓ itself in case of failure and thus has no externality to correct, while a monopolist overshoots, since $b^M > \ell$.

2.3 Lending capacity and the overhang threshold

Whether a firm with an inherited debt D_0 is able to obtain outside financing depends on two constraints: the resources it must raise at $t = 1$, and the maximum amount that can be recovered by the lender, established in Lemma 1. The shock affects the first constraint. A fall in ϵ reduces the income the firm has available without affecting the original debt, D_0. When the required resources exceed the recovery ceiling, no competitor will lend. Debt overhang is thus a region of the parameter space generated by the model, rather than an assumption.

The resources the firm must raise depend on how the inherited claim is treated when it borrows elsewhere. We make the following assumption,

Assumption 2 (Seniority)**.** Inherited debt D_0 is senior to any new lending. Switching lenders accelerates the incumbent's claim, so that D_0 falls due at $t = 1$ in cash rather than being pledged against date-2 income.

Assumption 2 is what distinguishes the incumbent from a competitor. Without the acceleration assumption, a competitor could lend against date-2 income while leaving the old claim in place, and the incumbent's bargaining position would be no different from any other creditor's. Under Assumption 2, the firm's resources at $t = 1$ are its revenue ϵy_0 and any advance A, which can be used to pay for the cost of the project X and the original accelerated claim D_0. Feasibility therefore requires

$$\epsilon y_0 + A \;\geq\; X + D_0 \qquad \Longrightarrow \qquad A \;\geq\; X + D_0 - \epsilon y_0. \tag{5}$$

Therefore, a lender that advances A against repayment b' breaks even only if $V(b') \geq RA$. Since Lemma 1 bounds V above by $V(b^M)$, the most that can be recovered by a lender requires $RA \;\leq\; V(b') \;\leq\; V(b^M)$, or equivalently,

$$A \;\leq\; \frac{V(b^M)}{R} = \frac{1 + 6\ell + \ell^2}{8R}. \tag{6}$$

No lender can advance more than this in present value. Notice that the ceiling in (6) is a consequence of risk shifting. A lender that wished to lend a larger amount would need to require a higher repayment, above b^M, which would push the borrower into gambles that repay less in expected value.

Combining the budget constraint in (5) with the debt ceiling in (6) yields a threshold on inherited debt,

$$\bar{D} \equiv \epsilon y_0 - X + \frac{1 + 6\ell + \ell^2}{8R}. \tag{7}$$

A firm inheriting more than $\bar{D}$ is in debt overhang after the shock. The following lemma records what this implies for its access to credit.

Lemma 2 (No outside credit)**.** *If $D_0 > \bar{D}$, no lender can offer a contract that is both feasible for the firm and break-even for the lender. The incumbent therefore faces no competition.*

Lemma 2 shows that there is no competition in this setting. Moral hazard caps what any lender can recover, and when the inherited claim exceeds that limit, no contractual arrangement offered by a competitor bank can cover the firm's sunk obligations while leaving the competitor a profit. The firm is therefore locked in with the incumbent. At $D_0 = \bar{D}$, (5) and (6) hold with equality, leaving a single advance $A = V(b^M)/R$ at which the lender breaks even.

The locked-in result depends on refinancing being uncapped. Under Assumption 1, a competitor can retire the incumbent's claim without the diversion cap applying, and thus the largest advance it can make is $V(b^M)/R$ and the threshold $\bar{D}$ depends on this ceiling. If refinancing were capped at X along with the advance, no lender could provide more than X in total. The project cost would absorb it and would leave nothing for the inherited claim. In that case, the threshold would collapse to $\bar{D}' = \epsilon y_0$. Then the boundary of the overhang region would carry no trace of the frictions that generate it.

2.4 Three regions of inherited debt

Overhang is one of the three regions of the debt space. In this subsection we characterize these regions, which supply welfare benchmarks for Section 3.3.

At low levels of debt, the firm is the residual claimant, and its objective coincides with the planner's. A bank advancing A and collecting b earns $V(b) - RA$, which Bertrand competition drives to zero, so $RA = V(b)$ and the firm's date-2 payoff is

$$R\big(\epsilon y_0 + A - X - D_0\big) + U(b) = R\big(\epsilon y_0 - X - D_0\big) + W(b). \tag{8}$$

The first term does not depend on the contract, so the firm maximises total surplus. By Lemma 1 this implies $b = \ell$, which requires an advance of exactly

ℓ/R, since $V(\ell) = \ell$. Whether it can borrow depends on D_0. Define

$$D^{fb} \;\equiv\; \epsilon y_0 - X + \frac{\ell}{R} \;\; (< \bar{D})$$

as the maximum debt at which the first-best advance is affordable. The debt space is thus split at D^{fb} and at $\bar{D}$: below D^{fb} the firm borrows the surplus-maximising amount, and above $\bar{D}$ no second lender will finance it. In the intermediate region, $D_0 \in (D^{fb}, \bar{D}]$, the firm is forced to over-borrow.

Proposition 1 (Debt regions)**.** *Suppose banks compete and $\ell \le \epsilon y_0$, and let $A^{\min} \equiv \max\{0,\, X + D_0 - \epsilon y_0\}$ be the smallest feasible advance. Then:*

(i) if $D_0 \le D^{fb}$, the firm attains $b = \ell$ and the outcome is first-best, with risk $p^*(\ell) = \frac{1-\ell}{2}$ and project surplus $W(\ell)$;

(ii) if $D^{fb} < D_0 \le \bar{D}$, the firm must borrow $A^{\min} > \ell/R$, and the competitive repayment $\hat{b}$ solving $V(\hat{b}) = RA^{\min}$ satisfies $\ell < \hat{b} < b^M$, so that risk is excessive and worsening in D_0;

(iii) if $D_0 > \bar{D}$, no competitive contract exists and the incumbent is a monopolist.

The condition $\ell \le \epsilon y_0$ means that the firm is worth more alive than liquidated; it is carried forward as part of Assumption 3(ii) below.

The three regions differ in which constraint binds on the firm's financing. In region (i) the firm's obligations are low enough such that the surplus-maximising advance ℓ/R is feasible. The firm borrows exactly what it wants and the first-best allocation is achieved. In region (ii) the existing obligations exceed what ℓ/R would cover, so the firm must over-borrow, and by (2) the higher repayment implies a higher risk. Thus, $\hat{b}$ rises above ℓ and risk rises above the efficient level. The distortion grows as D_0 grows. In Region (iii), the required advance surpasses the ceiling (6) and outside lending stops. The firm is locked-in with the incumbent. Figure 3 plots the resulting repayment schedule.

In region (ii) the competitor lends the firm enough to retire the incumbent's original claim, which for D_0 near $\bar{D}$ far exceeds the project's cost. The competitor chooses to retire the original claim and fund the project in order to profit from it. The inherited claim is senior and accelerated; if it were not settled at $t = 1$, the firm would be liquidated before the project is carried out.

The remainder of the paper focuses on region (iii), debt overhang. Regions (i) and (ii) supply the welfare benchmark of zombie lending, spelled out in Section 3.3.

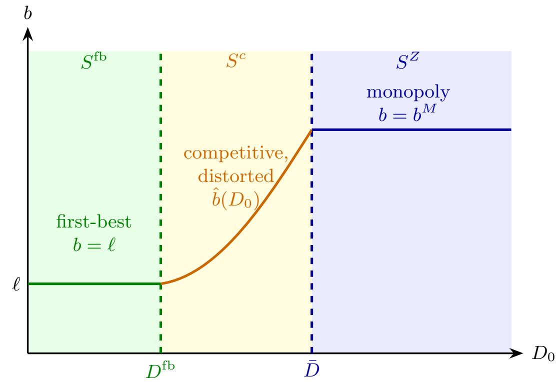


Figure: Equilibrium repayment b as a function of inherited debt D_0

Figure 3: The debt trichotomy. Equilibrium repayment as a function of inherited debt D_0, partitioned by the ceiling $V(b^M)$ that risk shifting places on what lenders can recover. For $D_0 \leq D^{fb}$ the firm receives the surplus-maximising repayment $b = \ell$ and the first-best is achieved. For $D^{fb} < D_0 \leq \bar{D}$ the firm over-borrows, so the competitive repayment $\hat{b}(D_0)$ rises above ℓ and risk becomes excessive. At $\bar{D}$ the competitive market shuts down and the incumbent implements b^M thereafter. Importantly, b is continuous at $\bar{D}$ but welfare is not. See Section 3.3.

2.5 Maintained assumptions

The debt overhang region is of interest only if the firm holds a project that is worth financing. The following three conditions, which we assume throughout, set out when that is the case.

Assumption 3 (Efficient and financeable project)**.** The project satisfies:

(i) the project is worth doing:

$$W(\ell) - RX \;\geq\; p_0\,\epsilon y_0 + (1-p_0)\ell; \tag{i}$$

(ii) the firm is viable, insolvent, and can cover the cost from income:

$$\max\{\ell, X\} \;\leq\; \epsilon y_0 \;\leq\; b^M; \tag{ii}$$

(iii) investment is incentive-compatible:

$$RX \;\leq\; U(b^M) = \frac{(1-\ell)^2}{16}. \tag{iii}$$

Condition (i) says the project is worth enacting: the surplus from the project must exceed the firm's outside option. The left-hand side, $W(\ell) - RX$, is the social surplus that the project generates at the first-best contract net of its funding cost. The right-hand side, $p_0\epsilon y_0 + (1-p_0)\ell$, is the value of the outside option: the full value of the BAU technology. The firm can either decline credit and run BAU unlevered, obtaining $p_0\epsilon y_0 + (1-p_0)\ell$, or accept the advance ℓ/R, put it in the risk-free technology by diverting it and obtain $\ell + p_0(\epsilon y_0 - \ell)$ at $t = 2$. The two coincide. Borrowing at the first-best rate and diverting therefore leaves the firm indifferent between borrowing and not.

Condition (ii) contains three inequalities. The first, $\ell \leq \epsilon y_0$, was already imposed in Proposition 1 but restated here for completeness. It states that the firm is worth more alive than liquidated. As a consequence, liquidation will be a dominated strategy for the banks in Section 3.2. The second, $\epsilon y_0 \leq b^M$, says BAU output is not enough to service the monopoly repayment. This is the precise sense in which the firm is insolvent, and as a consequence any firm that diverts its advance will go on to default. The third, $X \leq \epsilon y_0$, says the project costs less than the firm's income at $t = 1$. Therefore, if the monopolist wishes to fund investment, it will simply collect a lower payment from the firm and allow the internal funds to cover the project rather than extending fresh funds. [4]

Condition (iii) is the diversion constraint evaluated at the monopoly repayment. A firm holding X after settling its date-1 obligation earns $U(b^M)$ by investing and RX by diverting and storing. Thus, for investment to be carried out we need $U(b^M) \geq RX$. The BAU technology is not part of this tradeoff because a diverting firm is planning to default: It owes b^M at $t = 2$ but will produce only $\epsilon y_0 \leq b^M$. Therefore it defaults and retains nothing. The assumptions are jointly satisfiable.

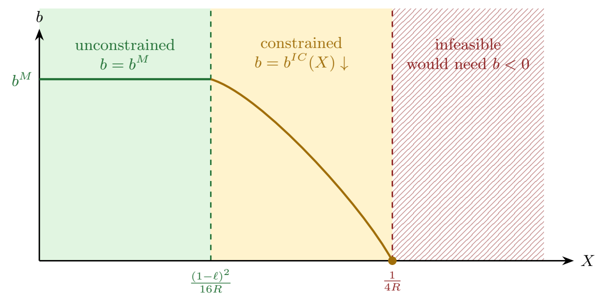
The cost regimes. The diversion constraint is $U(b) \geq RX$, and since U falls from $U(0) = \frac{1}{4}$, whether investment can be financed is characterized by a single function:

regime	monopolist implements	treated in
$RX \leq U(b^M) = \frac{(1-\ell)^2}{16}$	b^M, constraint slack	main analysis
$U(b^M) < RX \leq \frac{1}{4}$	$b^{IC} = 1 - 2\sqrt{RX}$	Section 3.7
$RX > \frac{1}{4}$	investment impossible	Appendix A

The bound $\frac{1}{4}$ is the largest value U can take, and hence the most the firm could ever gain from investing at any repayment. The temptation to divert is RX and

[4] This inequality does not follow from the first: condition (iii) caps X at $(1-\ell)^2/16R$, which for small ℓ exceeds ℓ, so X may be larger than the recovery.

Cost trichotomy: repayment the monopolist implements, continuous in X



(holding R, ℓ fixed; internal to the overhang region $D_0 > \bar{D}$)

Figure 4: The cost trichotomy. Repayment the monopolist implements as a function of the investment cost X, partitioned by the same ceiling $V(b^M)$. For $X \leq (1-\ell)^2/16R$ the diversion constraint is slack and the monopolist implements b^M. Beyond it the constraint binds, and the largest incentive-compatible repayment consistent with investment, $b^{IC}(X) = 1 - 2\sqrt{RX}$, falls continuously to zero at $X = 1/4R$, beyond which no repayment can deter diversion and investment is impossible at any contract. Section 3.7 derives b^{IC}.

grows linearly with the project's cost, whereas the reward is capped given that X buys access to a new technology rather than scaling the old one. The cap is itself also a consequence of risk shifting, since maximum output approaches 1 only through very risky gambles. Condition (iii) already forces $RX < (1-\ell)^2/16 < \frac{1}{4}$, so the third row lies outside the maintained assumptions.

Both partitions are generated by the same bound, $V(b^M)$. Risk shifting generates a ceiling on what any lender can recover. Read along the debt axis, that ceiling partitions D_0 into the three regions of Proposition 1 and Figure 3; read along the cost axis, it partitions RX into the three regimes of Figure 4. The paper's mechanism lives where the top debt region meets the bottom cost regime: the project is cheap enough to be financeable, and the firm is indebted enough that only the incumbent can finance it. Section 3 shows that the incumbent may prefer not to fund such a firm.

3 Zombie Lending

This section contains the paper's main result. Section 3.1 sets out the incumbent's problem and characterizes the potential equilibrium strategies. Section 3.2 ranks these strategies and shows that the equilibrium turns on a single inequality comparing what the bank can extract from a firm under its BAU technology against what it can recover from a firm carrying out an investment project. Sections 3.3 and 3.4 give the welfare cost and the comparative statics. The three subsections that follow examine the boundaries of the result: Section 3.5 treats the case of a firm with operational losses and Section 3.6 the case of an undercapitalized bank.

3.1 The incumbent's problem

Under Proposition 1, no outside contract is feasible if $D_0 > \bar{D}$. The incumbent, as a monopolist, offers a contract (r, b). Unlike the competitor, it does not need to settle D_0 at $t = 1$ in cash: the claim is already its own and D_0 is sunk and rolled forward, not a fresh outflow the contract must finance. The bank maximises expected profit subject to four constraints:

date-1 feasibility	$r \leq \epsilon y_0 - \mathbf{1}_I X,$	(C1)
date-2 feasibility	$b \leq$ output realised at $t = 2,$	(C2)
participation	firm's payoff $\geq 0,$	(C3)
no diversion	$U(b) \geq RX$ whenever funds are advanced.	(C4)

where $\mathbf{1}_I$ equals one if the contract funds the project and zero otherwise. Constraint (C4) is treated as loose in this section and as binding in Section 3.7.

The constraints sharply reduce the space of contracts, which is a priori potentially large. Profits are increasing in both r and b, thus the only real questions are whether the bank funds the project, and if not, whether it keeps the firm alive. That leaves four candidates stated in the following result:

Proposition 2 (Candidate strategies)**.** *Suppose $D_0 > \bar{D}$ and Assumption 3 holds. Then the incumbent's optimal offer lies in $\mathcal{S} = \{I, Z, ZP, L\}$:*

- (I) restructure*: (r^I, b^I) with $r^I = \epsilon y_0 - X$ and $b^I = b^M$;*
- (Z) zombie-lend*: (r^Z, b^Z) with $r^Z = b^Z = \epsilon y_0$, advancing nothing;*
- (ZP) partial zombie-lend*: (r^P, σ^ℓ) with $r^P = \epsilon y_0$;*
- (L) liquidate at once*: $(\sigma^\ell, 0)$.*

Under (I) the bank allows enough funds to fund the project, and sets the date-2 repayment to b^M. This is restructuring in the traditional sense: the inherited claim D_0 is retired and replaced by a new contract. The bank writes down part of what it was owed, since $r^I + V(b^M)/R = \bar{D}$, and in the overhang region the present value of the return from restructuring is lower than D_0 by construction. Under this contract, the firm retains $U(b^M) > 0$. The constraint that binds the monopolist is the firm's risk response: any repayment above b^M induces enough additional gambling to reduce the bank's own expected receipt, by Lemma 1.

Under (Z) and (ZP) the firm nets zero at both dates and nothing is advanced. The payment the bank collects falls short of the contractual obligation D_0. The arrangement resembles subsidised credit from outside. But D_0 is sunk: the firm cannot pay it under any contract and the bank collects the firm's entire cash flow. Under (ZP), the bank follows a strategy of forbearance-then-seizure: collecting BAU payments while it can and liquidating at $t = 2$. Zombie lending is thus a form of forbearance, in that the bank does not enforce its original claim but no debt is forgiven and the whole of the firm's operating revenue accrues to the lender.

The associated profits, in date-2 units, are

$$\Pi^Z = \underbrace{R\,\epsilon y_0}_{t=1,\text{ stored}} + \underbrace{p_0\,\epsilon y_0}_{t=2\text{ if BAU succeeds}} + \underbrace{(1-p_0)\ell}_{\text{if BAU fails}} = \epsilon y_0(R + p_0) + (1-p_0)\ell, \quad (9)$$

$$\Pi^{ZP} = R\,\epsilon y_0 + \ell, \quad (10)$$

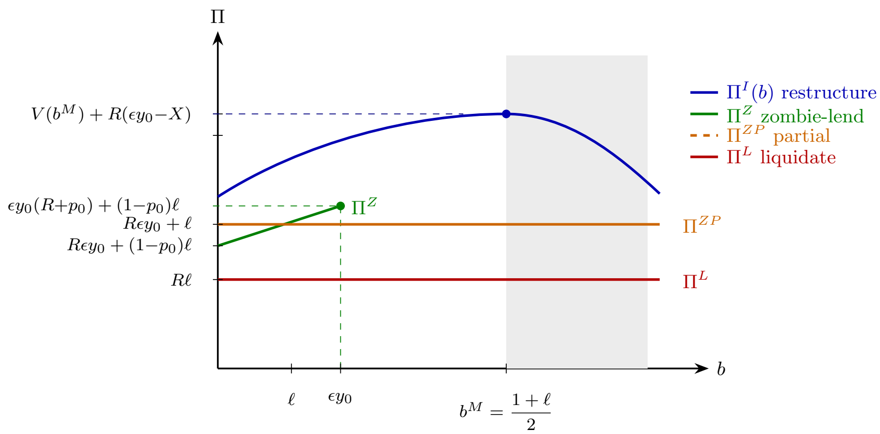
$$\Pi^L = R\ell, \quad (11)$$

$$\Pi^I = V(b^M) + R\big(\epsilon y_0 - X\big). \quad (12)$$

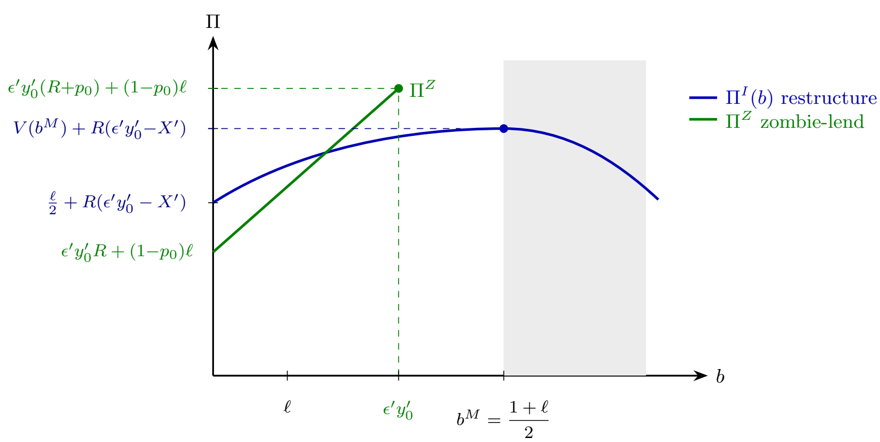
Under (Z), the bank collects three flows without any transfer to the firm. Under (ZP) the bank takes the date-1 income and liquidates at $t = 2$. Under (L) it liquidates early and stores the proceeds. Finally, under (I) it earns the expected receipt $V(b^M)$ on the restructured claim, net of the cost X so that the project is funded. Figure 5 plots the four schedules against b: risk shifting bends Π^I over at an interior maximum at b^M, while feasibility cuts Π^Z off at its endpoint.

3.2 Equilibrium

The equilibrium is determined in two steps. We first rank the three strategies that do not fund the project, since they differ only in when the bank stops collecting and takes the recovery. We then set the survivor strategies against restructuring, and that comparison is the paper's main result.



(a) Baseline: restructuring dominates.



(b) Higher X' and higher $\epsilon' y_0'$: zombie lending dominates.

Figure 5: Bank profit under each strategy, as a function of the date-2 repayment b. The restructuring schedule $\Pi^I(b) = V(b) + R(\epsilon y_0 - X)$ inherits the shape of V and peaks at b^M, since demanding more induces additional risk. The zombie-lending schedule $\Pi^Z(b) = R\epsilon y_0 + p_0 b + (1 - p_0)\ell$ rises linearly in b but is truncated at $b = \epsilon y_0$, where date-2 output has been exhausted. Thus it sets $b^Z = \epsilon y_0$. Condition (†) compares the two maxima. Panel (a) shows a case in which restructuring dominates and Panel (b) raises the project's cost to $X' > X$ lowering Π^I uniformly, and raises the firm's income to $\epsilon' y_0' > \epsilon y_0$, which lifts the zombie-lending schedule. The schedules Π^{ZP} and Π^L are flat and are dominated throughout whenever $\ell \leq \epsilon y_0$.

Proposition 3 (Hierarchy of non-funding strategies)**.** *Among* $\{Z, ZP, L\}$*,*

$$\Pi^{Z} - \Pi^{ZP} = p_0\big(\epsilon y_0 - \ell\big), \qquad \Pi^{ZP} - \Pi^{L} = R\,\epsilon y_0 - (R-1)\ell.$$

Hence (Z) *is optimal if and only if* $\epsilon y_0 \geq \ell$*;* (ZP) *if and only if* $\ell\frac{R-1}{R} \leq \epsilon y_0 < \ell$*; and* (L) *if and only if* $\epsilon y_0 < \ell\frac{R-1}{R}$*.*

The threshold $\epsilon y_0 = \ell$ separates (Z) from (ZP), and compares what the firm produces at $t = 2$ against what liquidation recovers. The bank keeps the firm alive to the end exactly when its output exceeds its scrap value. The threshold $\epsilon y_0 = \ell(R-1)/R$, which separates (ZP) from (L), compares the extra period of collection against the interest earned by liquidating early and storing the proceeds. Since $(R-1)/R$ is small at realistic rates, early liquidation requires income close to zero, and that region vanishes as $R \to 1$. Notice that Proposition 3 invokes none of the maintained assumptions and holds on their whole domain. Assumption 3(ii) places the firm on the upper side of the first threshold, so (Z) survives the ranking and the equilibrium reduces to a single comparison, which we state as follows.

Theorem 1 (Zombie lending)**.** *Suppose* $D_0 > \bar{D}$ *and Assumption 3 holds with* (C4) *slack* ($RX \leq (1-\ell)^2/16$)*. Then strategies* (ZP) *and* (L) *are dominated, and the equilibrium is* (Z) *if*

$$\underbrace{p_0\,\epsilon y_0 + (1-p_0)\ell + RX}_{\textit{value of not funding}} \;\geq\; \underbrace{V(b^M) = \frac{1+6\ell+\ell^2}{8}}_{\textit{most the loan can earn}} \tag{†}$$

and (I) *otherwise. At equality the bank is indifferent, and we select* (Z)*.*

Condition (†) is the paper's central object. The left-hand side collects what the lender forgoes by restructuring: the expected BAU repayment $p_0\epsilon y_0$, the recovery $(1-p_0)\ell$ it collects when BAU fails, and the funding cost RX it avoids by advancing nothing. The right-hand side is the maximum a loan can ever earn, given the borrower's risk shifting response.

The bank may prefer to zombie lend even if the firm has a project worth undertaking, by Assumption 3(i), which planner would fund. However, the bank compares the share of the project's surplus it can capture against the whole of the BAU cash flow it can collect. Risk shifting caps what the bank can capture from restructuring at $V(b^M)$, strictly below the surplus the project generates, since any attempt to claim more induces gambling that destroys the claim. Nothing caps what it can extract under zombie lending: its cash flow is given by the BAU technology, and the bank takes all of it.

The two frictions interact here, and neither suffices alone. Absent risk shifting, the right-hand side of (†) would be the project's full expected return, which under Assumption 3(i) is greater than the BAU revenue stream and thus the left-hand side. Therefore the bank would always fund the project. Absent debt overhang, a competitor would refinance the firm on the terms of Proposition 1, and the preferences of the incumbent would be irrelevant. Zombie lending requires both risk shifting, to cap what the incumbent can recover, and debt overhang, to ensure that no one else can bid for that firm. Figure 6 shows the resulting partition of firm characteristics.

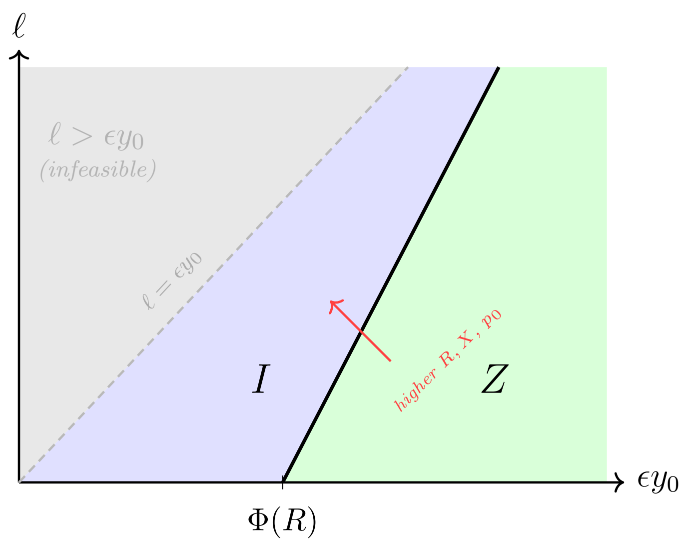


Figure 6: Equilibrium regions in the $(\epsilon y_0, \ell)$ plane. The frontier solves (†) with equality and has intercept $\Phi(R) = \left[V(b^M) - (1 - p_0)\ell - RX\right]/p_0$ on the income axis. To its right the bank zombie-lends (Z); to its left it restructures (I). In the shaded region, above the 45° line, the firm is worth more liquidated. The red arrow shows the direction of the comparative statics. See Section 3.4.

3.3 Welfare

Proposition 1 supplies two allocations against which to measure the outcome of Theorem 1. We state our welfare benchmarks.

Proposition 4 (Welfare ranking)**.** *In date-2 units, with* $C \equiv R(\epsilon y_0 - X - D_0)$

common to the first two allocations,

$$\underbrace{S^{fb} = C + W(\ell)}_{D_0 \leq D^{fb}} \geq \underbrace{S^c = C + W(\hat{b})}_{D^{fb} < D_0 \leq \bar{D}} > \underbrace{S^Z}_{zombie\ lending},$$

where $S^{fb} - S^c = \frac{1}{4}(\hat{b} - \ell)^2 \geq 0$ *and the second inequality is strict under Assumption 3(i).*

All three allocations are measured net of the inherited claim, so the ranking is unaffected by D_0 itself. What differs is whether the project is funded and, if so, at what risk level. The term C nets the project's cost out of date-1 income and is shared by S^{fb} and S^c. They differ only in the point at which W is evaluated, which is the overall level of debt. Under zombie lending no project is carried out, so S^Z and thus total surplus is the BAU technology's, $R\epsilon y_0 + p_0 \epsilon y_0 + (1 - p_0)\ell$, net of the claim.

Notice that between the first-best and the competitive allocation the project is undertaken in both cases, and what is lost is only that it is undertaken at the wrong risk level. The loss is second-order in the distortion, of size $\frac{1}{4}(\hat{b} - \ell)^2$, and vanishes smoothly as $\hat{b} \to \ell$. When debt is large enough that zombie lending arises, the project is not undertaken at all, and the entire net surplus $W(\ell) - RX$ over the BAU technology is completely lost. Overhang is therefore not simply a matter of degree that worsens as debt rises: it produces a discontinuity at $\bar{D}$. Section 4 studies the policy implications of this result.

3.4 Comparative statics

We now establish the comparative statics of the equilibrium, given by (†). Four parameters move the condition: the firm's income ϵy_0, the project's cost X, the reliability p_0 of the BAU technology, and the interest rate R. Differentiating,

$$\frac{\partial}{\partial(\epsilon y_0)} = p_0 > 0, \qquad \frac{\partial}{\partial X} = R > 0, \qquad \frac{\partial}{\partial p_0} = \epsilon y_0 - \ell \geq 0, \qquad \frac{\partial}{\partial R} = X > 0,$$

the third sign following from Assumption 3(ii). Intuitively, more income to collect, a costlier project, and a safer BAU technology each widen the zombie region: the first raises what zombie lending yields, the second raises what funding the project costs, and the third makes the collectable cash flow more reliable.

A higher interest rate also favours zombie lending. Since b^M is independent of R by Lemma 1, a change in the cost of funds does not alter what the bank can

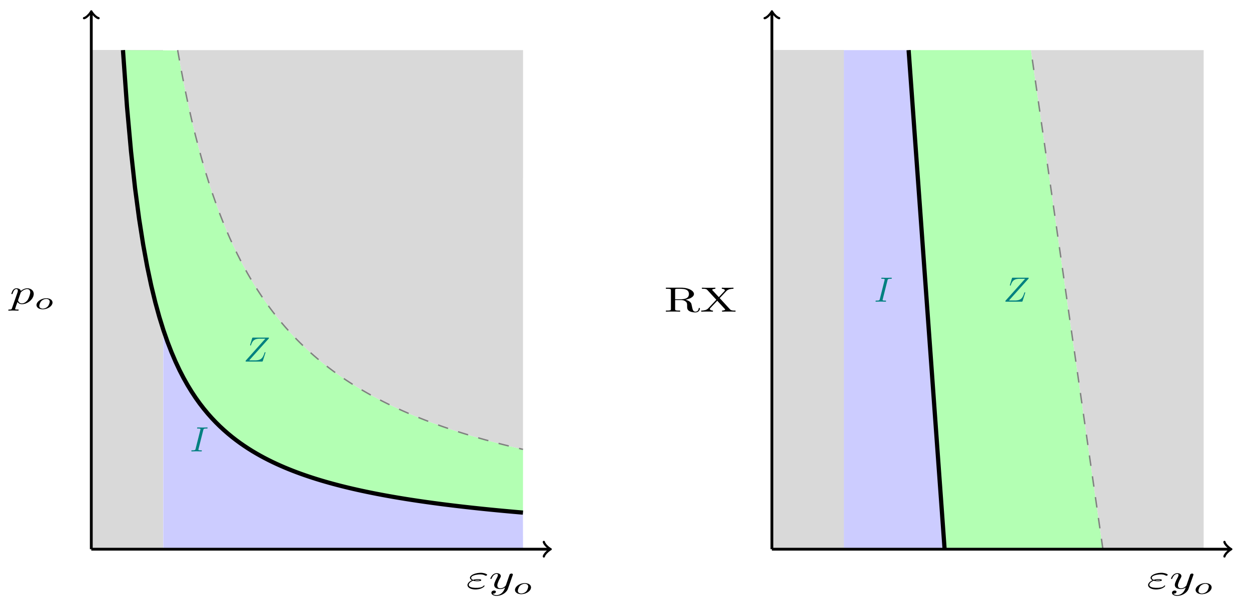


Figure 7: Equilibrium regions in two further planes. Left: in the $(\epsilon y_0, p_0)$ plane the frontier falls as p_0 rises. A firm whose existing technology is more reliable becomes a zombie at a lower level of income. Right: in the $(\epsilon y_0, RX)$ plane the frontier is linear with slope $-1/p_0$.

recover from a firm it finances. The channel through which it acts is by affecting the opportunity cost of leaving X with the firm. Rates therefore enter (†) only on the left hand side, and monotonically. Thus, if (†) holds at $R = 1$ it holds for every $R \geq 1$: tightening can push a firm into the zombie region, but loosening cannot pull it back out. Figure 7 shows the same frontier in two different planes, and Section 4 develops the policy consequences.

3.5 Operating losses

Throughout, the firm has been operationally viable but financially insolvent at the original level of debt. Suppose instead that it faces a fixed operating cost $\kappa > 0$ per date with $\kappa \geq \epsilon y_0$. The effect on the equilibrium is stated as follows.

Proposition 5 (No zombie lending with operating losses)**.** *If $\kappa \geq \epsilon y_0$, then (Z) and (ZP) are dominated by (L).*

Zombie lending is profitable only if there is income to extract. Keeping such a firm with operating losses afloat would require the bank to fund its losses, and a solvent bank does not choose to fund it.

The result delimits where the mechanism applies. Throughout the analysis so far, zombie firms have been financially distressed but operationally viable. The next subsection shows that the outcome can still arise when a firm makes operational losses if the bank is also distressed.

3.6 Undercapitalised banks

The empirical literature emphasises that zombie firms are more likely to exist when coupled with weakly capitalised banks. A weak bank cannot afford to recognise a loss, so liquidation becomes less attractive, widening the range of parameters for which it chooses to keep firms afloat. Within our model, that leaves untouched the comparison between zombie lending and restructuring, in which capital plays no part but does affect the incentives to liquidate.

Suppose the bank holds capital E against a regulatory minimum K, and faces a shock $\psi \sim G(\cdot)$ at $t = 1$. We assume that the bank faces a value $\nu < R\ell$ if its capital goes below that minimum. Liquidating the firm forces the bank to acknowledge the loan as non-performing and to write capital down to $E' \leq E$, leaving it more exposed when the shock arrives. Early liquidation is therefore worth

$$\Pi^L = R\ell\big[1 - G(E' - K)\big] + G(E' - K)\,\nu, \tag{13}$$

where the recovery value of the firm $R\ell$ is obtained if the bank survives the shock, and obtains ν if it does not. The remaining strategies are unaffected.

Proposition 6 (Undercapitalised banks)**.** *As $E \to K$:*

(a) under Assumption 3, the equilibrium of Theorem 1 is unchanged, and bank capital is irrelevant;

(b) if investment is unavailable, the early-liquidation region shrinks to $\epsilon y_0 < (u - \ell)/R$, and if $\nu \leq \ell$ the bank never liquidates early, so survival lending prevails everywhere;

(c) with the operating losses of Section 3.5, there exists $\bar{\nu} \equiv R(\epsilon y_0 - \kappa) + \ell \leq \ell$ such that the bank covers the firm's losses and liquidates late, (ZP), whenever $\nu \leq \bar{\nu}$.

Part (a) is an irrelevance result. This can be seen in (13): capital enters only Π^L, while by Theorem 1 the equilibrium comparison is between Π^Z and Π^I. A weak bank has a lower value of liquidation, which is an option it was not exercising to begin with. Thus, when a firm has a financeable project the equilibrium is the same whether the bank is strong or weak.

Parts (b) and (c) of the Proposition deal with firms for which liquidation was the best alternative. A bank near its regulatory minimum cannot afford to recognize losses, so the liquidation option loses value. Therefore, firms that a well-capitalized bank would liquidate, i.e., those with collapsed revenues in (b) or with operating losses in (c), instead receive survival lending. Such a bank lends not in expectation of recovery but because recognising the loss would jeopardise its own solvency. A financially distressed bank keeps such firms afloat but does not affect the incentive to restructure loans to viable firms with a profitable project.

3.7 Incentive-constrained restructuring

Theorem 1 assumed the diversion constraint (C4) to be slack at b^M, which is condition (iii) of Assumption 3. We now relax that assumption and take the middle row of the trichotomy in Section 2.5: the constraint binds, but investment remains implementable, albeit at a lower repayment. The main result of the paper continues to hold.

Constraint (C4) caps the repayment at the largest incentive-compatible level,[5]

$$b^{IC} \;\equiv\; 1 - 2\sqrt{RX}, \tag{14}$$

which falls as the project grows costlier. Two thresholds determine the regime. As RX exceeds $(1-\ell)^2/16$, this bound is below the monopolist's preferred repayment, b^M, and thus the diversion constraint, not risk shifting, determines the implementable repayment. As RX is below $(1-\epsilon y_0)^2/4$ the firm would owe more than its BAU output under diversion and it needs to default. The regime is therefore

$$\frac{(1-\ell)^2}{16} \;<\; RX \;\le\; \frac{(1-\epsilon y_0)^2}{4}, \tag{15}$$

non-empty exactly when the firm is insolvent in the sense of Assumption 3(ii).

Proposition 7 (Constrained restructuring)**.** *Suppose Assumption 3 holds and $RX > (1-\ell)^2/16$ so that (C4) binds. Then on regime (15):*

[5] Appendix A derives the following expressions, including continuity at $b^{IC} = b^M$, as part of the proof of Proposition 7.

(a) *the best investment contract sets* $r^I = \epsilon y_0 - X$ *and* $b = b^{IC}$*, with profit* $V(b^{IC}) + R(\epsilon y_0 - X)$*;*

(b) *every comparison in Theorem 1 and Proposition 3 holds with* $V(b^M)$ *replaced by* $V(b^{IC})$*;*

(c) $V(b^{IC})$ *strictly decreases in* X*, and* $V(b^{IC}) \to V(b^M)$ *as* $RX \downarrow (1-\ell)^2/16$.

Part (b) shows that the structure of the equilibrium is unchanged: only the value of the right-hand side of (†) falls, from $V(b^M)$ to $V(b^{IC})$. Part (c) shows that a costlier project damages restructuring through two different channels. First, it directly raises RX on the left hand side of (†), and second, it tightens the diversion constraint by (14), lowering the implementable repayment and shrinking $V(b^{IC})$ on the right. Both movements push toward zombie lending: the firms with the most to gain from investing are the least likely to be financed. Note that in this regime, the maximum repayment is capped below b^M for the monopolist, so the liquidation strategies are not necessarily dominated and the full hierarchy of Proposition 3 becomes relevant again.

4 Policy

Zombie lending in our model is a socially suboptimal decentralised equilibrium. We now discuss several policy implications that arise from the model and its mechanism. We consider three: monetary policy, bank recapitalisation, and debt haircuts.

Proposition 8 (Policy)**.** *Suppose* $D_0 > \bar{D}$ *and Assumption 3 holds. Then:*

(i) Zero pass-through. $b^M = \frac{1+\ell}{2}$ *does not involve* R.

(ii) Monetary policy. *The left-hand side of (†) increases in* R*, so a higher rate enlarges the zombie region, and if (†) holds at* $R = 1$ *it holds for all* $R \geq 1$*. The set on which no rate cut restores investment is* $\Omega_1 = \{\epsilon y_0 \geq \Phi\}$*, where*

$$\Phi(p_0, \ell, X) \;=\; \frac{1}{p_0}\big[\, V(b^M) - (1-p_0)\ell - X \,\big].$$

(iii) Recapitalisation. *Bank capital* E *enters only through the liquidation payoff; it never enters* Π^I*. Recapitalisation therefore acts solely on the liquidation margin and never on the investment margin.*

(iii.a) Well-capitalised bank. *Under Assumption 3, Π^L is dominated for all $D_0 > \bar{D}$, so the equilibrium is invariant to E.*

(iii.b) Undercapitalised bank ($E \to K$). *By Proposition 6, recapitalisation can move the equilibrium from survival lending, (Z) or (ZP), toward early liquidation (L), but never toward investment (I). When $\nu \leq \ell$ the early-liquidation region is empty and this margin too disappears.*

In neither case does recapitalisation restore the efficient project.

(iv) Haircuts. *Reducing D_0 below $\bar{D}$ restores competitive lending and below D^{fb} the first-best, but neither is privately optimal for the incumbent.*

Part (i) of the proposition states the zero pass-through result. In the model, the monopolistic power of the bank creates a wedge between the opportunity cost of funds and the relevant rate for the firm: the repayment a lender with market power demands is pinned by the borrower's risk response and not by the cost of funds. Thus, under debt overhang we obtain the zero pass-through result.

Part (ii) deals with monetary policy. Monetary policy usually works by lowering the opportunity cost of funds to induce investment. However, the rates in the model affect only the incumbent's opportunity cost of leaving funds X with the firm in a monotone way: a higher rate makes zombie lending more attractive, which is the result in (ii). Tightening can push a firm into the zombie region and loosening cannot pull it back out.

The set Ω_1 collects the configurations on which even a cut to $R = 1$ leaves (†) satisfied. In a way, it measures how much room monetary policy retains. Neither Ω_1 nor its complement is degenerate, as Appendix A verifies. The frontier moves with the parameters as follows,

parameter	$\partial\Phi/\partial\,\cdot$	
X	$-1/p_0 < 0$ always	costlier projects enlarge Ω_1
p_0	$\left[8X - (1-\ell)^2\right]/8p_0^2 < 0$ under (iii)	safer BAU enlarges Ω_1
ℓ	sign $= \text{sign}(p_0 - p^M)$	ambiguous

The first two rows say that the configurations in which monetary policy is completely ineffective are those in which zombie lending was most attractive to begin with. The entry for ℓ is ambiguous given that a higher recovery raises both what the lender collects when BAU fails, and $V(b^M)$. The effect that dominates depends on which technology fails more often.

Part (iii) shows that recapitalisation does not restore investment. When zombies arise at capitalised banks, recapitalizing has no effect: strengthening the bank raises the value of a strategy it was not exercising. A weakly capitalized bank cannot afford to recognise a loss, but capital acts only where liquidation is a live option, among firms with collapsed revenues or operating losses that are worth more liquidated than alive. Strengthening banks can induce liquidation of these firms but cannot restart investment.

Part (iv) deals with haircuts. Fundamentally, zombies arise because of the interaction between monopoly power under debt overhang and moral hazard that limits the willingness of the bank to extend new funds. Proposition 4 showed that the welfare space is partitioned: at low levels of debt the project is enacted at an efficient level of risk, at intermediate levels the project is enacted but at an inefficient level of risk, and at high levels the project is not carried out. Reducing the original debt D_0 below the overhang threshold $\bar{D}$ restores competitive access and captures the discrete gain, while reducing it below D^{fb} additionally removes the residual risk distortion. A debt haircut limits the capacity of the bank to extract funds from the firm and aligns the incentives for efficient risk-taking. A large enough haircut will increase social surplus and allow potential compensations so that everyone is better off. However, the incumbent does not find it privately optimal to do so. After a haircut of size ζ it collects at most $R(1-\zeta)D_0 \leq R\epsilon y_0$ and earns nothing on new competitive lending, against $\Pi^Z = R\epsilon y_0 + p_0\epsilon y_0 + (1-p_0)\ell$ from continuing to zombie-lend. Total surplus rises, so the firm gains more than the incumbent loses, and a transfer could in principle leave both better off. The transfer is not available privately: the firm's gain accrues as $U(b)$ at $t = 2$, and pledging it requires raising b, which is exactly what risk shifting prevents. The efficient restructuring is one the party holding the claim strictly prefers not to carry out, and one the beneficiary cannot pay for.

5 Discussion and Conclusion

Banks' lending to distressed firms at subsidized rates has been documented in several countries after the burst of bubbles. We explain why zombie-lending arises and continues to thrive even when renegotiation is possible, efficient projects are available, and banks are capitalized.

The model captures a fundamental conflict between lender and borrower in a double-decked incentive problem where the bank has monopoly power under debt overhang and the firm can divert funds or shift risk under moral hazard. The main drivers of this result are that the borrower becomes locked into a lending relationship with its incumbent bank and that it has access to a risk-shifting technology. Once the firm is in debt overhang, the incumbent bank can extract more funds from the firm than is socially optimal. Given the possibility of risk shifting, firms are not willing to use the fresh funds efficiently, and the bank anticipates this behavior. Previous explanations for delays in restructuring rely on distressed banks gambling for resurrection, strategic negotiation, or uncertainty. In the model presented, financially distressed firms are viable and have efficient projects, but these are not enacted when debt is large, even when the bank has funds. Moreover, the incentives of the actors do not allow the renegotiation process to solve the problem.

The mechanism present in the model suggests policy implications in stark contrast to previous papers and in line with several stylized facts. Partial debt forgiveness in the form of a debt haircut is shown to be privately suboptimal but necessary to restore investment. Fukuda and Nakamura (2011) shows that debt forgiveness was key in Japan. The model presented here explains why bank capitalization and monetary policy are not sufficient to restart investment as documented by Acharya et al. (2016) and Hoshi and Kashyap (2010). First, strengthening banks is insufficient if insolvency regimes are hostile to the reorganization of indebted firms, as it only attacks one side of the incentive problem. This model generates a typology of firms that may receive subsidized lending and the conditions under which a bank may decide to pursue this strategy. In previous explanations of debt forbearance, firms are insolvent. Efficiency requires the liquidation of these firms, a process which may be delayed due to an insolvent bank gambling for resurrection. Here, however, firms are financially insolvent but still have positive operational revenue and bank capital is irrelevant at the margin.

The problem is that firms' debt became suddenly large, and the firms that could still grow failed to do so because of that debt. Optimality requires that these firms restructure their debt, but banks refuse to do so at any level of capitalization. Second, the literature that highlights the ineffectiveness of monetary policy relies on the zero lower bound and uncertainty that causes cash hoarding (Ito and Mishkin, 2006; Krugman et al., 1998). We offer an alternative explanation. Lowering the policy rate decreases the opportunity cost for the bank, but not the effective rate facing the firm due to the debt overhang problem.

Additionally, there is only mixed evidence for the presence of risk shifting[6] and the literature has theoretically explored several mitigating factors.[7] The lending

[6]See De Jong and Van Dijk (2007), Eisdorfer (2008), and Gilje (2016) for evidence.

[7]See Almeida et al. (2011) and Barnea et al. (1980) for evidence.

mechanism we suggest offers an explanation for the lack of empirical risk shifting: because the banks can foresee the potential misuse of their funds, they can choose not to lend in the first place.

There are some institutional solutions that facilitate the financing of distressed firms, such as debt-equity swaps, DIP financing (Kahl, 2002), or the bank gaining an equity stake in the firm in a way that allows it to control the risk. These solutions can be understood within the model as requiring sizeable transaction costs or as entailing an acknowledgment of losses, and thus are addressed within the context of a financially distressed bank. Moreover, a bank may have limited managerial control over the firm it needs to take over.

Several assumptions are essential to these results. First, we assume that the shock is measure-zero. The choice of making the shock unexpected arises from the empirical fact that zombie firms tend to emerge after market bubbles burst. Alternatively, it can be argued that banks may also strategically restrain from taking risks even when a shock is expected (Nosal and Ordoñez, 2016). Second, we assume that there are two banks. This is done merely for simplicity, since the second bank plays the role of an alternative to the incumbent in normal times. This assumption can be easily relaxed to a continuum of banks. Third, in the model debt contracts are non-contingent. Regardless, as long as the bank's monitoring technology is not perfect, the results hold to a certain degree. Fourth, undercapitalized banks are modeled as being transitorily fragile. The main reason for this choice is tractability. The fact that banks normally smooth out losses over many periods using provisions before liquidating the firm justifies this assumption.[8] Fifth, we assume that the firm has debt with only one bank, thus abstracting from potential conflicts of interest between lenders. Notwithstanding, syndicated loans have been growing increasingly more common, and a large number of creditors can behave collusively.[9] Sixth, we assume that the firm cannot enter bankruptcy voluntarily or, equivalently, that remaining a zombie is preferable to bankruptcy. In a way, this model assumes a stigma associated with bankruptcy.[10] Finally, for the case of undercapitalized banks, some degree of bank opacity is implicitly assumed; otherwise investors would price the asset loss in the market value of the bank.[11]

[8] See for example Balboa et al. (2013) for evidence.

[9] See Gadanecz (2004) and Esty and Megginson (2003).

[10] Evidence of this can be found in Semadeni et al. (2008) and Gilson and Vetsuypens (1993).

[11] See BIS (2011), Flannery et al. (2004) and Huizinga and Laeven (2012) for evidence.

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A Proofs

Proof of Lemma 1. From (3), $V'(b) = \frac{1}{2}(1 - 2b + \ell)$ and $V''(b) = -1 < 0$, so V is strictly concave with maximum at $b^M = \frac{1+\ell}{2}$; substituting gives $p^M = \frac{1-\ell}{4}$ and $V(b^M) = \frac{1+6\ell+\ell^2}{8}$. From (4), $W'(b) = \frac{1}{2}(\ell - b)$, so W peaks at $b = \ell$, where $V(\ell) = \frac{\ell}{2}\big[(1-\ell)+(1+\ell)\big] = \ell$. Since $\ell < 1$, $b^M > \ell$ and $V(b^M) - \ell = \frac{1+6\ell+\ell^2-8\ell}{8} = \frac{(1-\ell)^2}{8} > 0$. □

Proof of Lemma 2. Any candidate contract must satisfy the date-1 budget (5) and the pledgeability ceiling (6), hence

$$X + D_0 - \epsilon y_0 \;\le\; A \;\le\; \frac{V(b^M)}{R},$$

which requires $D_0 \le \bar{D}$, contradicting $D_0 > \bar{D}$. □

Proof of Proposition 1. Part (iii) is Lemma 2. For (i)–(ii), Bertrand competition drives lender profit to zero, $V(b) = RA$. In date-2 units the firm's payoff is

$$R\big(\epsilon y_0 + A - X - D_0\big) + U(b) = \underbrace{R\big(\epsilon y_0 - X - D_0\big)}_{\text{contract-independent}} + V(b) + U(b),$$

so up to a constant the firm maximises $W(b)$: under competition the firm is residual claimant and its objective coincides with the planner's. Since W peaks at ℓ and $V(\ell) = \ell$, the target advance is $A = \ell/R$.

Feasible advances lie in $[A^{\min}, X + D_0]$ with $A^{\min} = \max\{0,\, X + D_0 - \epsilon y_0\}$: the floor is (5), and the ceiling follows from Assumption 1 together with the fact that paying the incumbent more than D_0 is a gift, not a settlement.

If $A^{\min} \leq \ell/R$, the firm borrows ℓ/R, the repayment is ℓ, and the outcome is first-best. The condition $A^{\min} \leq \ell/R$ is equivalent to $D_0 \leq \epsilon y_0 - X + \ell/R = D^{fb}$, and $D^{fb} < \bar{D}$ since $V(b^M) - \ell = \frac{(1-\ell)^2}{8} > 0$. (If instead $\ell/R > X + D_0$, then $D_0 < \ell/R - X < \epsilon y_0 - X$, so $A^{\min} = 0$ and the firm can decline credit and self-fund; an unlevered firm keeps the recovery itself, solves $\max_p \{p(1-p) + (1-p)\ell\}$, chooses $p = \frac{1-\ell}{2}$, and attains $W(\ell)$, so the first-best obtains regardless. The step $\ell/R < \epsilon y_0$ uses $\ell \leq \epsilon y_0$ together with $R > 1$.)

If $D^{fb} < D_0 \leq \bar{D}$, the floor binds: $A^{\min} = X + D_0 - \epsilon y_0 > \ell/R$. The competitive repayment solves $V(\hat{b}) = RA^{\min}$, i.e.

$$b^2 - (1+\ell)b + \big(2RA^{\min} - \ell\big) = 0, \qquad \Delta = 8\big(V(b^M) - RA^{\min}\big) > 0 \iff D_0 < \bar{D}.$$

Competition selects the smaller root, since a higher repayment on the same locus leaves positive profit and is undercut; hence $\hat{b} < b^M$. And $RA^{\min} > \ell = V(\ell)$ with V increasing on $[0, b^M)$ gives $\hat{b} > \ell$. Since $A^{\min}$ increases in D_0 and V is increasing below b^M, $\hat{b}$ increases in D_0, and risk $p^*(\hat{b})$ falls below the efficient level accordingly.

It remains to verify that the firm invests rather than deviating at the first-best contract. The constant $R(\epsilon y_0 - X - D_0)$ is common to every option and cancels. Declining credit and running BAU unlevered is worth $p_0 \epsilon y_0 + (1-p_0)\ell$. Accepting the advance ℓ/R and diverting leaves the firm levered at $b = \ell$, servicing ℓ from BAU output and retaining nothing on failure: $\ell + p_0(\epsilon y_0 - \ell)$. These coincide identically, and both are dominated by investment under Assumption 3(i). □

Proof of Proposition 2. Strategy (I). Profit $V(b) + Rr$ increases in r, so (C1) binds at $r^I = \epsilon y_0 - X \geq 0$ by Assumption 3(ii). Over b, profit peaks at b^M by Lemma 1. Constraint (C3) holds since the firm's payoff is $\epsilon y_0 - r^I - X +$

$U(b^M) = U(b^M) > 0$. For (C4): after paying r^I the firm holds X; investing yields $U(b^M)$, while diverting yields RX stored plus a BAU continuation of zero, since $\epsilon y_0 \le b^M$ implies default. Investment therefore requires $U(b^M) \ge RX$, which is Assumption 3(iii).

Strategy (Z). With no investment, (C1) gives $r \le \epsilon y_0$ and (C2) gives $b \le \epsilon y_0$. Profit increases in both, so $r^Z = b^Z = \epsilon y_0$. The firm's payoff is zero, equal to its outside option, so (C3) holds with equality; (C4) is vacuous, as nothing is advanced.

Strategy (ZP). The date-1 reasoning is identical, and the date-2 entry σ^ℓ is worth ℓ.

Completeness. Any r strictly between r^I and r^Z either funds the project while leaving idle cash, and is dominated by (I), which sweeps it; or fails to fund the project, and is dominated by (Z), which sweeps more. □

Proof of Proposition 3. Subtracting (10) from (9) gives $p_0(\epsilon y_0 - \ell)$, and (11) from (10) gives $R\epsilon y_0 - (R-1)\ell$. The boundaries are ordered, $\frac{R-1}{R} < \frac{R-1+p_0}{R+p_0} < 1$, the middle term being the (Z)-versus-(L) boundary; cross-multiplying yields $-p_0 < 0$ and $-1 < 0$ respectively. The three regions therefore partition. □

Proof of Theorem 1. By Assumption 3(ii), $\epsilon y_0 \ge \ell$, so Proposition 3 gives $\Pi^Z \ge \Pi^{ZP}$ and $\Pi^Z \ge \Pi^L$: zombie lending dominates the other extraction strategies, and the equilibrium is $\max\{\Pi^Z, \Pi^I\}$. Comparing (9) and (12),

$$\epsilon y_0(R + p_0) + (1 - p_0)\ell \;\ge\; V(b^M) + R(\epsilon y_0 - X),$$

and $R\epsilon y_0$ cancels from both sides, leaving (†). □

Proof of Proposition 4. The first two allocations differ only in the point at which W is evaluated, and (4) gives $W(\ell) - W(b) = \frac{1}{4}(b - \ell)^2$; setting $b = \hat{b}$ yields the stated gap, non-negative and zero only at $\hat{b} = \ell$. For the second inequality, under zombie lending no project is funded, so $S^Z = R\epsilon y_0 + p_0\epsilon y_0 + (1 - p_0)\ell - RD_0$. Subtracting from S^{fb} and cancelling $R(\epsilon y_0 - D_0)$ leaves $W(\ell) - RX - p_0\epsilon y_0 - (1 - p_0)\ell$, which is non-negative by Assumption 3(i) and strict when that condition holds strictly. □

Proof of Proposition 6. As $E \to K$, $G(E' - K) \to 1$, so (13) gives $\Pi^L \to \nu$.

(a) By Theorem 1 the equilibrium comparison is between Π^Z and Π^I, neither of which involves Π^L. Lowering Π^L cannot change a comparison it does not enter.

(b) With (I) unavailable, apply Proposition 3 with $\Pi^L = \nu$: $\Pi^{ZP} \geq \nu$ iff $R\epsilon y_0 + \ell \geq \nu$, i.e. $\epsilon y_0 \geq (\nu - \ell)/R$, a bound that is non-positive when $\nu \leq \ell$. The (Z)-versus-(ZP) boundary $\epsilon y_0 = \ell$ is unchanged.

(c) With operating losses $\Pi^{ZP} = R(\epsilon y_0 - \kappa) + \ell$ and, as in Proposition 5, $\Pi^Z \leq \Pi^{ZP}$. The bank prefers (ZP) to liquidating iff $\nu \leq R(\epsilon y_0 - \kappa) + \ell \equiv \bar{\nu}$, which is at most ℓ since $\epsilon y_0 \leq \kappa$. □

Proof of Proposition 5. Keeping the firm alive requires covering $\epsilon y_0 - \kappa \leq 0$ each period, so

$$\Pi^Z = R(\epsilon y_0 - \kappa) + p_0(\epsilon y_0 - \kappa) + (1 - p_0)\ell, \qquad \Pi^{ZP} = R(\epsilon y_0 - \kappa) + \ell.$$

Then $\Pi^{ZP} - \Pi^Z = p_0\big[\ell - (\epsilon y_0 - \kappa)\big] > 0$ since $\epsilon y_0 - \kappa \leq 0 < \ell$; and $\Pi^L - \Pi^{ZP} = (R-1)\ell - R(\epsilon y_0 - \kappa) \geq 0$, both terms being non-negative and the second strict when $\kappa > \epsilon y_0$. □

Proof of Proposition 7. Derivation of (14) *and* (15). Constraint (C4) states $U(b) \geq RX$, which by (3) is $(1-b)^2/4 \geq RX$, or $b \leq b^{IC} \equiv 1 - 2\sqrt{RX}$. The constraint binds at the monopoly optimum when $b^{IC} < b^M$, that is when $RX > (1-\ell)^2/16$; at equality $b^{IC} = b^M$, so the characterisation is continuous at the boundary of Assumption 3(iii). For the diverting firm to default, as the argument for condition (iii) required, we also need $b^{IC} \geq \epsilon y_0$, or $RX \leq (1 - \epsilon y_0)^2/4$. Combining gives (15), which is non-empty precisely when $\epsilon y_0 < b^M$.

(a) On regime (15) we have $b^{IC} \geq \epsilon y_0$, so a diverting firm defaults and (C4) reduces to $b \leq b^{IC}$. Since $b^{IC} < b^M$ and V is increasing on $[0, b^M)$ by Lemma 1, the constrained optimum is $b = b^{IC}$. The firm's payoff is $U(b^{IC}) = RX > 0$; at the binding constraint it is indifferent, and we select investment.

(b) Expressions (9)–(11) involve neither X nor the investment contract. Substituting b^{IC} into (12) and cancelling $R\epsilon y_0$ gives the result.

(c) $db^{IC}/dX = -R/\sqrt{RX} < 0$ and $V'(b) > 0$ for $b < b^M$, so $dV(b^{IC})/dX < 0$. Continuity at $RX = (1-\ell)^2/16$, where $b^{IC} = b^M$, gives the limit. □

Proof of Proposition 8. (i) is Lemma 1. For (ii), $\partial(\text{LHS of } (\dagger))/\partial R = X > 0$ while the right-hand side is independent of R; setting $R = 1$ and solving $(\dagger)$ for ϵy_0 gives Φ. For (iii), E enters no payoff other than the liquidation branch: the advances and repayments determining Π^Z and Π^I are independent of E, so $\partial \Pi^I/\partial E = 0$ and recapitalisation never moves the investment margin. *(iii.a)* When the bank is well capitalised, Theorem 1 gives $\Pi^L < \max\{\Pi^Z, \Pi^I\}$ for all

$D_0 > \bar{D}$; since E enters only Π^L, the equilibrium is invariant to E, and by Proposition 3 the set on which (L) is optimal, $\{\epsilon y_0 < \ell(R-1)/R\}$, is empty at $R = 1$. *(iii.b)* When $E \to K$, raising E raises the survival weight on liquidation and hence the relative value of (L); by Proposition 6(b)–(c) this can convert (Z) or (ZP) into (L) but leaves Π^I unchanged, so (I) is never induced. Under $\nu \leq \ell$ the region in Proposition 6(b) is empty and early liquidation does not occur. For (iv), the surplus gains are Proposition 4. For private suboptimality, after a haircut of size ζ the incumbent collects at most $R(1-\zeta)D_0 \leq R\epsilon y_0$ and earns zero on new competitive lending, whereas $\Pi^Z = R\epsilon y_0 + p_0\epsilon y_0 + (1-p_0)\ell > R\epsilon y_0$. $\square$